

CAN THE SIERPINSKI GRAPH BE EMBEDDED IN THE HAMMING GRAPH?

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*In memory of my father,
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1910-2005*

Dad was thrown out of his high school algebra class for asking questions such as, "What is algebra?" and "What do 'x' and 'y' mean?" He never went back. But he loved manipulable puzzles. I was about six when he showed me how to untangle the Chinese Rings (a physical manifestation of the Sierpinski graph).

ABSTRACT. The (generalized & expanded) Sierpinski graph, $S(n, m)$, and the Hamming graph, K_m^n , have the same set of vertices, $\{0, 1, \dots, m-1\}^n$. The edges of both are (unordered) pairs of vertices. Each set of edges is defined by a different property so that neither contains the other. We ask if there is a subgraph of K_m^n isomorphic to $S(n, m)$ and show that the answer is yes. The recursively defined embedding map, $\varphi : S(n, m) \rightarrow K_m^n$, leads to a number of variations and ramifications. Among them is a simple algebraic formula for the solution of the Tower of Hanoi puzzle.

1. INTRODUCTION

1.1. Background. The standard architecture (connection graph) for a multiprocessor computer is the n -dimensional cube with $n = 6$ or 7 (so the number of processors is 64 or 128). Recently, the "Sierpinski gasket pyramid graph" has been proposed by William, Rajasingh, Rajan & Shantakumari [11] as an alternative (to the n -cube) . After determining the elementary graph-theoretic properties of the Sierpinski gasket pyramid graph, W-R-R&S proposed studying its "message routing and broadcasting" properties. This is the first of several papers following up on that suggestion.

Our goal with this paper has been to learn more about the structure of Sierpinski graphs. Our goal for the next paper is to solve the edge-isoperimetric problem (*EIP*) on the (generalized & expanded) Sierpinski graph, $S(n, m)$. $S(n, m)$ has the same vertex set as K_m^n , the Hamming graph (see below for definitions), but fewer edges, so it was natural to ask if $S(n, m)$ could be embedded in K_m^n . If so, it would provide a context for $S(n, m)$ which could lead to insight. The structure of K_m^n is relatively transparent, its *EIP* has been solved (see [3], p. 112) and it is closely related to the n -cube (the Sierpinski gasket pyramid graph's competition for multiprocessor architecture). Embedding $S(n, m)$ into K_m^n turned out to be an interesting and challenging problem in its own right and its solution has unexpected ramifications.

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1.2. Definitions.

1.2.1. *Graphs.* An ordinary graph, $G = (V, E)$ consists of a set V , of vertices and a set $E \subseteq \binom{V}{2}$, of pairs of vertices called edges.

Example 1. K_m , the complete graph on m vertices has $V_{K_m} = \{0, 1, 2, \dots, m-1\}$ and $E_{K_m} = \binom{V_{K_m}}{2}$.

Example 2. The (disjunctive) product, $K_m \times K_m \times \dots \times K_m = K_m^n$ is called the Hamming graph. Two vertices (n -tuples of vertices of K_m) have an edge between them if they differ in exactly one coordinate (i.e. are at Hamming distance 1).

1.2.2. The (Generalized & Expanded) Sierpinski Graph, $S(n, m)$. $S(n, m)$, $n \geq 1$, $m \geq 2$ was defined in 1944 by Scorer, Grundy and Smith [10]: $V_{S(n, m)} = \{0, 1, \dots, m-1\}^n$. For $\{u, v\} \in \binom{V_{S(n, m)}}{2}$, $\{u, v\} \in E_{S(n, m)}$ iff $\exists h \in \{1, 2, \dots, n\}$ such that following 3 conditions hold:

- (1) $u_i = v_i$ for $i = 1, 2, \dots, h-1$;
- (2) $u_h \neq v_h$; and
- (3) $u_j = v_h$ and $v_j = u_h$ for $j = h+1, \dots, n$.

The motivation for defining $S(n, m)$ was that $S(n, 3)$ is the graph of the 3-peg Tower of Hanoi puzzle with n disks [10]. Scorer, Grundy and Smith also pointed out that the graph defined by topologist Wacław Sierpinski in 1915, known as the Sierpinski gasket graph, is a contraction of $S(n, 3)$ where every edge of $S(n, 3)$ not contained in a triangle (K_3) is contracted to a vertex. The Sierpinski gasket pyramid (aka the Sierpinski sponge), is a 3-dimensional analog of the Sierpinski gasket, and its graph is a similar contraction of $S(n, 4)$. Jakovac [6] generalized the construction to $S[n, m]$, the contraction of $S(n, m)$ in which every edge of $S(n, m)$ not contained in a K_3 is contracted to a vertex. He worked out some properties of $S[n, m]$ such as hamiltonicity and chromatic number ($\chi(S[n, m]) = m$).

It seems apparent that the nomenclature of these structures is a muddle. First came the Sierpinski gasket, a topological curiosity, and then the family of graphs ($S[n, 3]$) that comprise the boundary of the "gasket" at the n^{th} stage of its construction. The "gasket", a 2-dimensional structure, was generalized to m dimensions by replacing the equilateral triangle that Sierpinski started with, by an m -simplex. If $S[n, 3]$ is the Sierpinski (gasket) graph, then $S[n, m]$ should be called the "generalized Sierpinski graph". Scorer, Grundy & Smith noted that $S[n, 3]$ could be obtained from the Tower of Hanoi graph, $S(n, 3)$, by contracting certain edges to a single vertex. Conversely, $S(n, 3)$ is derived from $S[n, 3]$ by expanding certain vertices into an edge (and two vertices) and $S(n, m)$ is a generalization of that construction. For many purposes $S(n, m)$ is easier to deal with (than $S[n, m]$) because its vertices and edges are easily defined. So we propose calling $S(n, m)$ the "generalized & expanded Sierpinski graph" or "Sierpinski graph" for short (when the meaning is clear from context).

2. STRUCTURE OF $S(n, m)$

2.1. **The Basics.** $|V_{S(n, m)}| = m^n$. All $v \in V_{S(n, m)}$ have $m-1$ "interior" neighbors. These are the n -tuples that agree with v in all coordinates except the n^{th} (the case $h = n$ in the definition of edges in $S(n, m)$). If $v \neq i^n$ then v has one other ("exterior") neighbor: If $v \neq i^n = (i, i, \dots, i)$, then $\exists h, 1 \leq h < n$, such that

$v_h \neq v_{h+1} = v_{h+2} = \dots = v_n$ and by definition the exterior neighbor of v is $u = v_1 v_2 \dots v_{h-1} v_{h+1} v_h v_h \dots v_h$ (note that this relationship between u and v is symmetric). Thus i^n , with $i = 0, 1, 2, \dots, m-1$, has degree $m-1$ and every other vertex has degree m . Summing the degrees of all vertices we get $m(m-1) + (m^n - m)m = m^{n+1} - m$. Since each edge is incident to two vertices, $|E_{S(n,m)}| = (m^{n+1} - m)/2$.

The vertices that agree in all except the last coordinate induce a complete subgraph, K_m . There are m^{n-1} such K_m 's, nonoverlapping and containing all the vertices of $S(n, m)$. This constitutes a K_m -decomposition of $S(n, m)$. Since any vertex is incident to at most one exterior edge, any triangle (K_3) must contain at least two internal edges. But then the third edge would also be internal to the same K_m , so our K_m -decomposition is unique. The vertices i^n , for $i = 0, 1, 2, \dots, m-1$, are called *corner* vertices of $S(n, m)$.

Example 3. $S(n, 2)$ is just a path of length 2^n . Its endpoints are the two corner vertices, 0^n & 1^n . Every other vertex is incident to two edges, one coming from (the direction of) 0^n and the other going toward 1^n . That it is a single path (and not a path plus disjoint cycles) is shown by the observation that the two vertices, u, v , connected by the edge $\{u, v\}$ are binary representations of two consecutive integers. One of them, say u , has $u_h = 0$ and the other $v_h = 1$. The last $n-h+1$ components of u contribute $2^{n-h-1} + 2^{n-h-2} + \dots + 1 = 2^{n-h} - 1$ to its binary representation whereas that of v contributes 2^{n-h} .

Figure 1 shows diagrams of $S(n, 2)$ for $n = 1, 2, 3$ with coordinates.

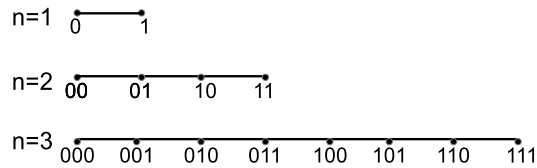


Figure 1- $S(n, 2)$ for $n = 1, 2, 3$

Figure 2 shows diagrams of $S(n, 2)$ for $n = 1, 2, 3$ with coordinates.

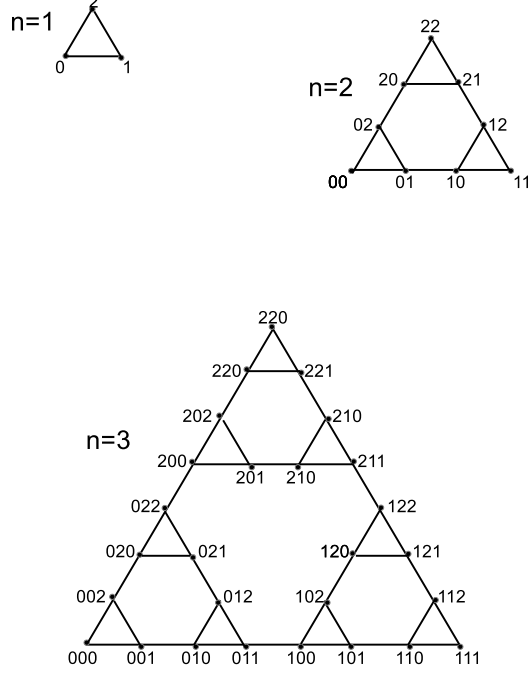


Figure 2- $S(n, 3)$ for $n = 1, 2, 3$

2.2. Symmetry of $S(n, m)$.

Theorem 1. (Theorem 4.14 of [5]) *The symmetry group of $S(n, m)$ is \mathcal{S}_m , the symmetric group on m generators.*

Proof. Let π be any permutation of $\{0, 1, 2, \dots, m-1\}$, and let it act on the vertices of $S(n, m)$ in the obvious way,

$$\sigma_\pi(v) = \sigma_\pi(v_1, v_2, \dots, v_n) = (\pi(v_1), \pi(v_2), \dots, \pi(v_n)).$$

If $\{u, v\} \in E_{S(n, m)}$ then

$$\begin{aligned} \sigma_\pi(u) &= \sigma_\pi(u_1, u_2, \dots, u_n) \\ &= \sigma_\pi(u_1, u_2, \dots, u_{h-1}, u_h, u_{h+1}, u_{h+1}, \dots, u_{h+1}), \text{ where } u_h \neq u_{h+1}, \\ &= (\pi(u_1), \pi(u_2), \dots, \pi(u_{h-1}), \pi(u_h), \pi(u_{h+1}), \pi(u_{h+1}), \dots, \pi(u_{h+1})) \end{aligned}$$

and

$$\begin{aligned} \sigma_\pi(v) &= \sigma_\pi(v_1, v_2, \dots, v_n) \\ &= \sigma_\pi(v_1, v_2, \dots, v_{h-1}, v_h, v_{h+1}, v_{h+1}, \dots, v_{h+1}) \\ &= \sigma_\pi(u_1, u_2, \dots, u_{h-1}, u_{h+1}, u_h, u_h, \dots, u_h) \\ &= (\pi(u_1), \pi(u_2), \dots, \pi(u_{h-1}), \pi(u_{h+1}), \pi(u_h), \pi(u_h), \dots, \pi(u_h)). \end{aligned}$$

So $\{\sigma_\pi(u), \sigma_\pi(v)\} \in E_{S(n, m)}$ and σ_π is a symmetry of $S(n, m)$.

Conversely, suppose σ is a symmetry of $S(n, m)$. Any symmetry, σ , of $S(n, m)$ must take corners to corners so $\sigma(i^n) = j^n$ for some $j \in \{0, 1, \dots, m-1\}$. We

then define a permutation on $\{0, 1, \dots, m-1\}$ by $\pi_\sigma(i) = j$. We claim that when this action is extended as above, we get σ back again: By induction on h (in the definition of external edges). For $h = 0$, $\sigma(i^n) = j^n = \pi_\sigma(i)^n$. For $h = 1$, if $\sigma(ij^{n-1}) = (k_1, k_2, \dots, k_n)$ then since the subgraph $\{i\} \times S(n-1, m)$ contains i^n , its image under σ must contain $\pi_\sigma(i)^n$ and be $\{\pi_\sigma(i)\} \times S(n-1, m)$. Therefore $k_1 = \pi_\sigma(i)$. $\sigma(ij^{n-1})$ must be a corner vertex of $\{\pi_\sigma(i)\} \times S(n-1, m)$ which means that $(k_2, \dots, k_n) = k^{n-1}$ for some k . Since $ij^{n-1} \in \{i\} \times S(n-1, m)$ and $ji^{n-1} \in \{j\} \times S(n-1, m)$ are neighbors, $\sigma(ij^{n-1}) \in \{\pi_\sigma(i)\} \times S(n-1, m)$ and $\sigma(ji^{n-1}) \in \{\pi_\sigma(j)\} \times S(n-1, m)$ are also neighbors, so $k = \pi_\sigma(j)$ and $\sigma(ij^{n-1}) = \pi_\sigma(i) \pi_\sigma(j)^{n-1}$. The pattern for higher h is the same, so $\sigma(v_1 v_2 \dots v_n) = \pi_\sigma(v_1) \pi_\sigma(v_2) \dots \pi_\sigma(v_n)$ and the symmetry induced by π_σ is σ . \square

3. EMBEDDING $S(n, m)$ IN K_m^n

The vertex-sets of $S(n, m)$ and K_m^n are the same, $\{0, 1, \dots, m-1\}^n$. However the edge-sets are different: $E_{K_m^n} \not\subseteq E_{S(n, m)}$ because

$$|E_{K_m^n}| = n(m-1)m^n/2 > (m^{n+1} - m)/2 = |E_{S(n, m)}|.$$

Also $E_{S(n, m)} \not\subseteq E_{K_m^n}$ since the exterior edges of $S(n, m)$ differ in two or more coordinates. The question of whether $S(n, m)$ can be embedded in K_m^n is a special case of the subgraph isomorphism problem, well-known for its complexity (it is NP-complete [2]). However, we now show that such an embedding exists. First we consider a couple of special cases.

3.0.1. *The Case $m = 2$.* We showed in Example 1 that $S(n, 2)$ is a path from 0^n to 1^n . So the question of embedding of $S(n, 2)$ into $K_2^n = Q_n$, the graph of the n -cube, is the graph-theoretic classic, "Does Q_n contain a Hamiltonian path (a path passing through each vertex exactly once). The answer is yes (there are many, the Gray code being one (see Wikipedia on "Gray code")).

3.0.2. *The Case $n = 2$ (Giving $S(n, m)$ a twist).* Define $\tilde{S}(n, m)$ as the graph with

$$V_{\tilde{S}(n, m)} = \{0, 1, \dots, m-1\}^n$$

and

$$E_{\tilde{S}(n, m)} = \bigcup_{h=1}^n \left\{ \left\{ v^{(h-1)} i k^{n-h}, v^{(h-1)} j k^{n-h} \right\} : i \neq j \text{ \& } i + j = k \pmod{m} \right\}$$

where $v^{(h)} \in \{0, 1, \dots, m-1\}^h$. Note that $|V_{\tilde{S}(n, m)}| = |V_{S(n, m)}|$ and $|E_{\tilde{S}(n, m)}| = |E_{S(n, m)}|$. Also note that $\tilde{S}(n, m)$ is a subgraph of K_m^n (since $V_{\tilde{S}(n, m)} = V_{K_m^n}$ and the pairs of vertices that comprise an edge of $\tilde{S}(n, m)$ differ in exactly one coordinate, making it an edge of K_m^n). However, this does not prove that $S(n, m)$ is embeddable as a subgraph of K_m^n .

Lemma 1. $S(2, m) \cong \tilde{S}(2, m)$.

In the next section we will prove a more general theorem. It would seem natural to extend Lemma 1 to show that $\tilde{S}(n, m) \cong S(n, m)$. After many attempts to do so, we ran up against the following counterexample: Consider the vertex $(0, 1, 1)$ in $\tilde{S}(3, 3)$. It shares an edge with the vertices $(0, 1, 0)$ & $(0, 1, 2)$ ($h = 3$) as well as the

vertices $(0, 0, 1)$ ($h = 2$) and $(1, 1, 1)$ ($h = 1$). So $(0, 1, 1)$ has degree 4 in $\tilde{S}(3, 3)$, but the maximum degree of any vertex in $S(3, 3)$ is 3.

3.0.3. A Recursive Definition of $S(n, m)$. The Sierpinski graph, $S(n, m)$, has been defined analytically in Section 1.2.2. However, $S(n, m)$ may also be characterized recursively: $S(1, m) = K_m$ and given $S(n-1, m)$ for $n-1 \geq 1$, we construct $S(n, m)$ by making m copies, $\{i\} \times S(n-1, m)$, $0 \leq i < m$, and throwing in all edges of the form $\{ij^{n-1}, ji^{n-1}\}$ such that $0 \leq i, j < m$ and $i \neq j$. Note that every copy of $S(n-1, m)$ is connected to every other copy and the corners of $S(n, m)$ are the constant vertices i^n , $0 \leq i < m$.

3.1. The General Case (Giving $S(n, m)$ another Twist). The Scorer-Grundy-Smith definition of the (generalized and extended) Sierpinski graph may be regarded as a representation or coordinatization of the abstract graph. Our question of whether $S(n, m)$ may be embedded in K_m^n is equivalent to asking whether there is another definition of that abstract graph; a definition in which the set of vertices, $\{0, 1, \dots, m-1\}^n$, is the same, but the pairs of vertices, $\{u, v\}$ that constitute an edge differ in exactly one coordinate. An obvious candidate was $\tilde{S}(n, m)$. That worked for $n = 2$ (Lemma 1) but failed for $n = 3$.

Theorem 2. *There exists another coordinatization of $S(n, m)$ (call it $\tilde{\tilde{S}}(n, m)$) that is a subgraph of K_m^n . I.e. the edges of $\tilde{\tilde{S}}(n, m)$ consist of pairs of vertices that differ in exactly one coordinate.*

Proof. By induction on n : It is trivial for $n = 1$ ($\tilde{\tilde{S}}(1, m) = K_m = S(1, m)$). Assume that the theorem is true for $n-1 \geq 1$. Implicit in the statement of the theorem is the existence of a function, $\varphi^{(n)} : \{0, 1, \dots, m-1\}^n \rightarrow \{0, 1, \dots, m-1\}^n$ that takes each vertex of $S(n, m)$ to its coordinates in $\tilde{\tilde{S}}(n, m)$. We write this as $\varphi^{(n)}(v_1 v_2 \dots v_n) = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n)$. We have represented $S(n, m)$ recursively as

$$\{0, 1, \dots, m-1\} \times S(n-1, m)$$

(the disjoint union of m copies of $S(n-1, m)$), along with the external edges,

$$\{ij^{n-1}, ji^{n-1}\}, i, j \in \{0, 1, \dots, m-1\} \& i \neq j.$$

By the inductive hypothesis we have $\varphi^{(n-1)} : S(n-1, m) \rightarrow \tilde{\tilde{S}}(n-1, m)$ taking $v \in V_{S(n-1, m)}$ to its coordinates in $\tilde{\tilde{S}}(n-1, m)$. For each $i \in \{0, 1, \dots, m-1\}$ we define

$$\varphi_i^{(n-1)} : \{i\} \times S(n-1, m) \rightarrow \{i\} \times \tilde{\tilde{S}}(n-1, m)$$

by

$$\varphi_i^{(n-1)}(ij^{n-1}) = (i, \varphi^{(n-1)}((i+j) \bmod m)^{n-1}).$$

The ij^{n-1} are only the corner vertices of $\{i\} \times S(n-1, m)$, but as we showed in Theorem 1, any permutation of the underlying set, $\{0, 1, \dots, m-1\}$ (and here we are applying the permutation $j \rightarrow (i+j) \bmod m$), extends to a unique symmetry of $S(n-1, m)$. That extension constitutes $\varphi_i^{(n-1)}$. Now we define

$$\varphi^{(n)} = \bigoplus_{i=0}^{m-1} \varphi_i^{(n-1)},$$

the disjoint union of the component maps, $\varphi_i^{(n-1)}$, $0 \leq i \leq m-1$. We claim then that $\varphi^{(n)} : S(n, m) \rightarrow \tilde{S}(n, m)$ is the isomorphism we have been seeking: Edges internal to $\{i\} \times S(n-1, m)$ are preserved because $\varphi_i^{(n-1)} : \{i\} \times S(n-1, m) \rightarrow \{i\} \times \tilde{S}(n-1, m)$ is the composition of two isomorphisms $j \rightarrow (i+j) \pmod{m}$ and $\varphi^{(n-1)}$. By induction on n , the vertices incident to an edge internal to $\{i\} \times \tilde{S}(n-1, m)$ differ in exactly one coordinate.

The external edges, $\{ij^{n-1}, ji^{n-1}\}$, are preserved because

$$\begin{aligned} & \varphi^{(n)}(\{ij^{n-1}, ji^{n-1}\}) \\ &= \{\varphi^{(n)}(ij^{n-1}), \varphi^{(n)}(ji^{n-1})\} \\ &= \{\varphi_i^{(n-1)}(ij^{n-1}), \varphi_j^{(n-1)}(ji^{n-1})\} \\ &= \left\{ \left(i, \varphi^{(n-1)}(((i+j) \pmod{m})^{n-1}) \right), \left(j, \varphi^{(n-1)}(((j+i) \pmod{m})^{n-1}) \right) \right\} \\ &\in E_{\tilde{S}(n, m)} \text{ since } (i+j) = (j+i). \end{aligned}$$

Thus $\varphi^{(n)}(ij^{n-1})$ and $\varphi^{(n)}(ji^{n-1})$ differ in exactly one coordinate (the first). \square

Figure 3 shows $\tilde{S}(n, 3)$, $n = 1, 2, 3$.

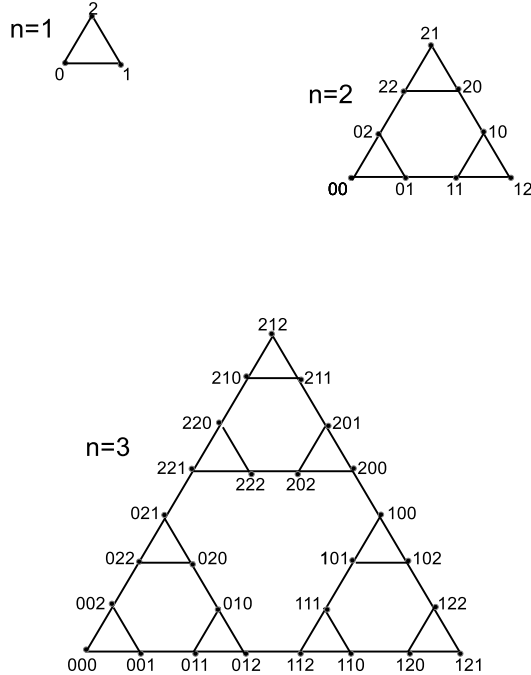


Figure 3- $\tilde{S}(n, 3)$ for $n = 1, 2, 3$

Note that in the case $n = 2$, $\varphi^{(1)}$ is the identity on $S(1, m)$ so $\tilde{S}(2, m) = \tilde{S}(2, m)$ and our proof includes Lemma 1. Geometric intuition had indicated that to embed $S(n, m)$ into K_m^n we had to give the copies of $S(n-1, m)$ a twist, which led to

the definition of $\tilde{S}(2, m)$. The counterexample showed that just one twist was not adequate for $S(3, m)$, that we had to give the copies of $\tilde{S}(2, m)$ another twist, leading to the definition of $\tilde{\tilde{S}}(3, m)$, *etc.*

4. COMMENTS AND CONCLUSIONS

4.1. Application to Layout Problems. Theorem 2 gives the solution of a wire-length problem. In laying out a graph, G , on a graph, H , edges of G must be assigned to paths with distinct endpoints. Since the shortest such path is a single edge, an embedding of G into H achieves the minimum wirelength (the sum of the lengths of all paths assigned to edges of G) and bandwidth (the maximum length of any path assigned to an edge of G) of any layout. Thus $WL(S(n, m) \rightarrow K_m^n) = |E_{S(n, m)}| = (m^{n+1} - m)/2$ and $BW(S(n, m) \rightarrow K_m^n) = 1$. In a sequel to this paper we hope to present a solution of the bandwidth problem for laying out $S(n, m)$ on a path, P_{m^n} , of length m^n . The problems of laying out $S(n, m)$ on a product of paths, $P_{m^{n_1}} \times P_{m^{n_2}} \times \dots \times P_{m^{n_k}}$ such that $n_1 + n_2 + \dots + n_k = n$ so as to minimize wirelength or bandwidth appears very challenging and will probably require new methods.

4.2. Relative Density of Edges. Since $S(n, m) \simeq \tilde{\tilde{S}}(n, m)$, a subgraph of K_m^n , one may ask about the density of $E_{\tilde{\tilde{S}}(n, m)}$ in $E_{K_m^n}$:

$$\begin{aligned} \frac{|E_{\tilde{\tilde{S}}(n, m)}|}{|E_{K_m^n}|} &= \frac{(m^{n+1} - m)/2}{(m^n(m-1)n/2)} \\ &= 1/n, \end{aligned}$$

which is arbitrarily small as n becomes large.

4.3. In Retrospect. Our initial guess, that $\tilde{\tilde{S}}(n, m)$ is isomorphic to $S(n, m)$, proved to be false in general, but we retained it in this presentation (for $n = 2$) because

- (1) The counterexample for $n = 3$ & $m = 3$ shows the subtlety of the problem, and
- (2) The proof for $n = 2$ gave the solutions $\tilde{v}_1 = v_1$, $\tilde{v}_2 = (v_1 + v_2) \bmod m$ of the recurrence inherent in the proof of Theorem 2. This suggested extending the formulas to $\tilde{v}_k^{(n)}$ as a function of (v_1, v_2, \dots, v_n) . The resulting formulas are in the next section.

4.4. Formulas for \tilde{v}_n . In the proof of Theorem 2 we have shown that

$$\tilde{v}_1^{(n)} = \varphi_1^{(n)}(v_1, v_2, \dots, v_n) = v_1$$

and for $1 < i \leq n$, $\tilde{v}_i^{(n)}$

$$\begin{aligned} \tilde{v}_i^{(n)} &= \varphi_i^{(n)}(v_1, v_2, \dots, v_n) \\ &= \varphi_{i-1}^{(n-1)}((v_1 + v_2) \bmod m, (v_1 + v_3) \bmod m, \dots, (v_1 + v_i) \bmod m). \end{aligned}$$

Lemma 2. $\forall n \geq i, \tilde{v}_i^{(n)} = \tilde{v}_i^{(i)}$

Proof. By induction on i . For $i = 1$, $\varphi_1^{(n)}(v_1, v_2, \dots, v_n) = v_1 = \varphi_1^{(1)}(v_1)$. Assume the Lemma is true for $i - 1 \geq 1$. Then if $n > i > 1$,

$$\begin{aligned}
 & \varphi_i^{(n)}(v_1, v_2, \dots, v_n) \\
 (4.1) \quad &= \varphi_{i-1}^{(n-1)}((v_1 + v_2) \bmod m, (v_1 + v_3) \bmod m, \dots, (v_1 + v_i) \bmod m) \\
 (4.2) \quad &\text{by Section 4.4,} \\
 &= \varphi_{i-1}^{(i-1)}((v_1 + v_2) \bmod m, (v_1 + v_3) \bmod m, \dots, (v_1 + v_i) \bmod m) \\
 &\quad \text{by the inductive hypothesis,} \\
 &= \varphi_i^{(i)}(v_1, v_2, \dots, v_i) \\
 &\quad \text{by Section 4.4 again.}
 \end{aligned}$$

□

So, to express $\tilde{v}_i^{(n)}$ as a function of v_1, v_2, \dots, v_n , we need only do it for $\tilde{v}_n^{(n)} = \tilde{v}_n$. Henceforth we shall drop the superscript, (n) , if the domain is clear from context.

Theorem 3. $\tilde{v}_n = \varphi_n(v_1, v_2, \dots, v_n) = \left(v_n + \sum_{i=1}^{n-1} 2^{n-1-i} v_i \right) \bmod m$.

Proof. By induction on n . It is trivially true for $n = 1$. Assume it is true for $n \geq 1$. Then

$$\begin{aligned}
 & \varphi_{n+1}(v_1, v_2, \dots, v_{n+1}) \\
 &= \varphi_n((v_1 + v_2) \bmod m, (v_1 + v_3) \bmod m, \dots, (v_1 + v_{n+1}) \bmod m) \\
 &\quad \text{by Section 4.4,} \\
 &= \left((v_1 + v_{n+1}) + \sum_{i=1}^{n-1} 2^{n-1-i} (v_1 + v_{i+1}) \right) \bmod m \\
 &\quad \text{by the inductive hypothesis,} \\
 &= \left(v_{n+1} + \left(v_1 + \sum_{i=1}^{n-1} 2^{n-1-i} v_1 \right) + \sum_{i=1}^{n-1} 2^{n-1-i} v_{i+1} \right) \bmod m \\
 &= \left(v_{n+1} + 2^{n-1} v_1 + \sum_{i=1}^{n-1} 2^{(n+1)-1-(i+1)} v_{i+1} \right) \bmod m \\
 &= \left(v_{n+1} + \sum_{i=1}^{(n+1)-1} 2^{(n+1)-1-i} v_i \right) \bmod m.
 \end{aligned}$$

□

4.5. **Inverting** $\varphi(v_1, v_2, \dots, v_n) = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n)$.

Theorem 4. $\varphi_i^{-1}(\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n) = \left(\tilde{v}_i - \sum_{j=1}^{i-1} \tilde{v}_j \right) \bmod m$.

Proof.

$$\begin{aligned}
(\varphi^{-1} \circ \varphi)_{i,j} &= \varphi_i \cdot \varphi_j^{-1} \\
&= \left(\overbrace{2^{i-2}, 2^{i-3}, \dots, 1, 1, 0, \dots, 0}^i \right) \cdot \left(\overbrace{0, 0, \dots, 0, 1, -1, \dots, -1}^j \right) \\
&= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \iota_{i,j}
\end{aligned}$$

□

Note that this inversion is actually valid over the integers, \mathbb{Z} (not just \mathbb{Z}_m), and for $n = \infty$.

When $m = 2$ these formulas reduce to

$$\widetilde{\widetilde{v}}_n = (v_n + v_{n-1}) \bmod 2$$

and

$$v_n = \left(\sum_{j=1}^n \widetilde{\widetilde{v}}_j \right) \bmod 2.$$

The embedding of $S(n, 2)$, a path of length 2^n , into $K_2^n = Q_n$, the graph of the n -cube, given by Theorem 2, turns out to be exactly the Gray code mentioned in Section 3.0.1. The result is trivial for $n = 1$, $\varphi^{(1)}$ being the identity. If it is the Gray code for $n - 1 \geq 1$, then by the recurrence at the beginning of Section 4.4, $\widetilde{\widetilde{v}}_i^{(n)}(v_1, v_2, \dots, v_n)$ will be the same for the first half of $S(n, 2)$ as it was for $S(n-1, 2)$ since $v_1 = 0 = \widetilde{\widetilde{v}}_1$. For the second half, we add $v_1 = 1$ to each component except the first, thereby complimenting it and reversing the order. That is the (recursive) definition of the Gray code.

4.6. The Answer to an Old Question. More than fifty years ago, we noticed the similarity between the natural (base 2) code that assigns to each n -tuple, v , of 0s and 1s the integer, $\eta(v)$, it represents in base 2, *i.e.*

$$\eta(v) = \sum_{i=1}^n 2^{n-i} v_i,$$

and the Gray code (See Wikipedia). The Gray code is defined recursively as a circular order on the vertices of the n -cube. The natural binary code can also be described recursively. But can the Gray code be represented by a formula, similar to that for $\eta(v)$? It seemed that some of the coefficients in any such formula,

$$\gamma(v) = \sum_{i=1}^n c_i v_i,$$

would have to be negative. But then the formula would take negative, as well as positive values, which presents a problem. We tried various ways to get around this difficulty but without success. However, the desired formula drops right out of our Example 3 and Section 4.5. In Example 3 we observed that $S(n, 2)$ is a path, with the vertices appearing in their natural (base 2) order. At the end of Section 4.5 we

noted that $\varphi^{(n)} : S(n, 2) \rightarrow \tilde{S}(n, 2) \subset K_2^n = Q_n$ carried $S(n, 2)$ to the Gray code. With the inverse of $\varphi_n : \{0, 1\}^n \rightarrow \{0, 1\}^n$ we have our formula

$$\begin{aligned} \gamma(\tilde{v}) &= \eta\left(\varphi^{-1}(\tilde{v})\right) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^i \tilde{v}_j \pmod{2} \right) 2^{n-i}. \end{aligned}$$

The essence of this relationship between η and γ is in the Wikipedia article "Gray Code" but presented in pseudocode rather than an algebraic formula.

4.7. Reverse Engineering. In reviewing the literature on Sierpinski graphs, we were chagrined to realize that a principle case of our title's question already had an answer: The (generalized & extended) Sierpinski graphs originated in work on the Tower of Hanoi puzzle [10]. A position of the Tower of Hanoi puzzle with n discs may be represented by $v \in \{0, 1, 2\}^n$, $v_i = j$ meaning that disc i is on peg j . The position is uniquely determined by the n -tuple since the discs on a peg must be stacked smaller on larger (the radius is decreasing in i). When disc i is moved from peg j_1 to j_2 , all discs on j_1 & j_2 must be larger than disc i . Thus the smaller discs must all be stacked on the third peg, $k \neq j_1, j_2$. When such a move is made, only one coordinate of the representing n -tuple, v , changes (v_i goes from j_1 to j_2). So what we have done here is reverse engineer the work of Scorer, Grundy & Smith in that case. However, even in that case there are interesting differences.

4.8. The Tower of Hanoi Graph. Let $T(n)$ be the Tower of Hanoi graph with 3 pegs and n discs. As described in the previous paragraph, vertices represent positions and edges represent the moving of a disc from one peg to another. Scorer, Grundy & Smith showed that $T(n)$ and $S(n, 3)$ are different coordinatizations of the same abstract graph. Therefore $T(n)$ and $\tilde{S}(n, 3)$ are isomorphic and both are subgraphs of K_3^n . So, is $T(n, 3) = \tilde{S}(n, 3)$?

Figure 4 shows $T(n)$, $n = 1, 2, 3$.

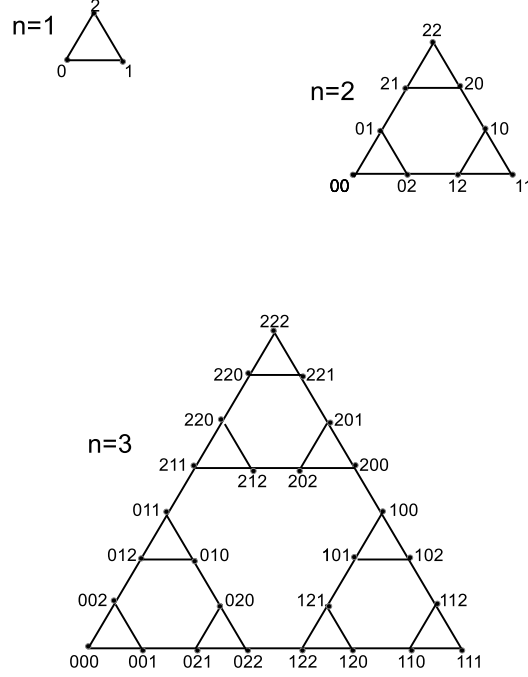


Figure 4- $T(n)$ for $n = 1, 2, 3$

A glance at diagrams of $\tilde{S}(n, 3)$ & $T(n)$ shows they are not equal. Already for $n = 2$ some coordinates are different. Is it possible to change our definition of φ a bit so that it takes $S(n, 3)$ to $T(n)$?

$\tau : S(n, 3) \rightarrow T(n)$ is defined recursively by the following slight variant of $\varphi : S(n, 3) \rightarrow \tilde{S}(n)$.

$$\tau^{(1)} = \iota_{K_3}, \text{ the identity on } K_3,$$

$$\text{For } n > 1, \tau_i^{(n)}(ij^{n-1}) = \left(i, \tau^{(n-1)}\left((2(i+j) \bmod 3)^{n-1}\right)\right),$$

$$\text{and } \tau^{(n)} = \bigoplus_{i=0}^{m-1} \tau_i^{(n)}.$$

The only difference is the factor of 2 multiplying $(i+j) \bmod 3$. The diagram for $T(2)$ differs from that for $\tilde{S}(2, 3)$ by an interchange $1 \longleftrightarrow 2$ in the second coordinate. This is due to the fact that when the large disc is moved from peg 0, the small disc must be on peg 1 or peg 2 and then the large disc must go on the other peg (2 or 1). Since $2 \times 0 = 0 \pmod{3}$, $2 \times 1 = 2 \pmod{3}$ and $2 \times 2 = 1 \pmod{3}$, multiplying $(i+j)$ by $2 \pmod{3}$ exactly represents this move.

4.9. The Structure of τ . As with φ , τ may be viewed as a nonsingular linear transformation $\tau(v_1, v_2, \dots, v_n) = (t_1, t_2, \dots, t_n)$. In particular,

$$t_1 = \tau_1(v_1, v_2, \dots, v_n) = v_1$$

and for $1 < i \leq n$,

$$\begin{aligned} t_i &= \tau_i^{(n)}(v_1, v_2, \dots, v_n) \\ &= \tau_{i-1}^{(n-1)}(2(v_1 + v_2) \bmod 3, 2(v_1 + v_3) \bmod 3, \dots, 2(v_1 + v_i) \bmod 3). \end{aligned}$$

And, as before (Lemma 2 and then Theorem 3), we have

Lemma 3. $\forall n \geq i, \tau_i^{(n)} = \tau_i^{(i)}$,

which may be proved in exactly the same way. So, to express t_i as a function of v_1, v_2, \dots, v_n , we need only do it for $\tau_n^{(n)} = t_n$.

Theorem 5. $t_n = \tau_n^{(n)}(v_1, v_2, \dots, v_n) = 2^{n-1} \left(v_n + \sum_{i=1}^{n-1} 2^{n-1-i} v_i \right) \bmod 3$, so $t_n = 2^{n-1} \widetilde{v}_n \pmod{3}$.

And, also as before (Theorem 4),

Theorem 6. $\tau_n^{-1}(t_1, t_2, \dots, t_n) = \left(2^{n-1} t_n - \sum_{j=1}^{n-1} 2^{j-1} t_j \right) \bmod 3$.

Again, the proofs of these theorems are essentially the same as for φ except we need the fact that $2^2 = 1 \pmod{3}$ for Theorem 6. $\tau^{-1} : T(n) \rightarrow S(n, 3)$ is actually the original Scorer-Grundy-Smith transformation.

The Tower of Hanoi is a fundamental example of a recursive algorithm. As such it is assigned as an exercise in programming courses. The theorems above provide a "hack" (illicit shortcut) for that exercise: In $S(n, 3)$ the shortest path from 0^n to 1^n is $S(n, 2)$, whose vertices appear in lexicographic order. So

$$\tau \circ \eta^{-1}(\ell)$$

gives the solution of the classical Tower of Hanoi puzzle, moving discs from 0^n to 1^n in $2^n - 1$ steps. More specifically, the i^{th} coordinate of the ℓ^{th} position is

$$\tau_i^{(n)}(\ell) = 2^{i-1} \left(\sum_{j=1}^{i-1} 2^{i-1-j} \ell_j + \ell_i \right) \bmod 3$$

where

$$\ell = \sum_{h=1}^n 2^{n-h} \ell_h, \ell_h = 0 \text{ or } 1.$$

(See, Dad, algebra can solve the Tower of Hanoi puzzle, and evidently the Chinese Rings too!). Clifford Wolfe came up with a similar formula for $2\tau_i^{(n)}(\ell)$ (<http://www.clifford.at/hanoi/>) by analysing the moves of individual discs:

$$2\tau_i^{(n)}(\ell) = ((i \bmod 2) + 1) \left\lfloor \frac{\ell + 2^i}{2^{i+1}} \right\rfloor \bmod 3.$$

4.10. Generalization of the Embeddings $\varphi^{(n)}, \tau^{(n)}$. Implicit in the recursive definitions of φ and τ are $m + 1$ permutations $\pi_i \in \mathcal{S}_m$, $0 \leq i \leq m$, : Let $\Pi_n = (\pi_0, \pi_1, \dots, \pi_m)$ then

$$\varepsilon_{\Pi_n} : S(n, m) \rightarrow K_m^n$$

is defined recursively by

$$\varepsilon_{\Pi_n}^{(1)} = \iota_{K_m}, \text{ the identity on } K_m,$$

and given

$\varepsilon^{(n-1)} : S(n-1, m) \rightarrow K_m^{n-1}$, $\varepsilon_{\Pi_n, i}^{(n)} : (i, S(n-1, m)) \rightarrow (\pi_m(i), K_m^{n-1})$ is defined by

$$\varepsilon_{\Pi_n, i}^{(n)}(ij^n) = \left(\pi_m(i), \varepsilon^{(n-1)}((\pi_i(j))^n) \right) \text{ and then}$$

$$\varepsilon_{\Pi_n}^{(n)} = \bigoplus_{i=1}^m \varepsilon_{\Pi_n, i}^{(n)}.$$

Theorem 7. *If $\varepsilon^{(n-1)}$ is an embedding and $\forall i, j < m$, $\pi_i(j) = \pi_j(i)$, then $\varepsilon_{\Pi_n}^{(n)}$ is an embedding.*

Example 4. *For $\varphi^{(n)}$, $\pi_m = \iota_{K_m}$ and for $i, j < m$, $\pi_i(j) = (i+j) \bmod m$. For $i < m$, $\pi_i(j) = \pi_j(i)$ since $i+j = j+i$.*

Example 5. *For $\tau^{(n)}$, $\pi_m = \iota_{K_m}$ and for $i, j < m$, $\pi_i(j) = 2(i+j) \bmod m$ (m must be odd in order for $j \rightarrow 2(i+j) \bmod m$ to be a permutation).*

Note that in each of these examples $\pi_m = \iota_{K_m}$. We may always assume, by way of normalizing to eliminate redundancy, that $\pi_m = \iota_{K_m}$. If $\pi_m \neq \iota_{K_m}$, the symmetry induced by π_m^{-1} will put it in that form.

Thus for each $\varepsilon^{(n-1)}$ we can take $\pi_i(j)$ to be $c(i+j) \bmod m$, where $\gcd(c, m) = 1$. There are $\phi(m)$ ways (ϕ is the Euler totient function) to choose c , so the number of nonisomorphic embeddings $\varepsilon^{(n)} : S(n, m) \rightarrow K_m^n$ we have constructed is $(\phi(m))^n$. In the words of Hardy & Wright, the order of $\phi(n)$ is "always 'nearly n '" (See Wikipedia, "Euler's Totient Function (Growth of the Function)") so for any $\delta > 0$, for m sufficiently large there are at least $m^{(1-\delta)n}$ such embeddings.

4.11. Constant Corners Property. One of the striking features of $S(n, m)$ & $T(n)$ is that corners (vertices of degree $m-1$) are constant (i^n for some i , $0 \leq i < m$).

In $\tilde{S}(n, 3)$ there is only one constant corner, 0^n , the two other corners alternating 0 & 1. Also, our mapping $\tau : S(n, 3) \rightarrow T(n)$ actually preserves the coordinates of corners ($\tau(i^n) = i^n$). Can $T(n) = T(n, 3)$ be generalized to $T(n, m)$, a subgraph of K_m^n , isomorphic to $S(n, m)$ and with constant corners? The recursive definition of $\tau : S(n, 3) \rightarrow K_3^n$ that characterizes $T(n)$ as its range, may be extended to $\tau : S(n, m) \rightarrow K_m^n$ for any odd m by letting $c = 2^{-1} \pmod{m}$ in the previous section. Then

$$\tau_i^{(n)}(ij^{n-1}) = \left(i, 2^{-1}\tau^{(n-1)}\left(((i+j) \bmod m)^{n-1} \right) \bmod m \right)$$

since multiplication by 2 is invertible \pmod{m} and $2^{-1} = \frac{m+1}{2} \pmod{m}$ (note that $2^{-1} = 2 \pmod{3}$, so it made no difference that we used 2 instead of 2^{-1} in Section 4.8, since $m = 3$ there). By induction, the $T(n, m)$ so defined has constant corners since $2^{-1}(i+i) = i \pmod{m}$. However, the definition does not work for even

m because 2 does not have a multiplicative inverse mod m . $\tilde{S}(n, 2)$ trivially has constant corners, but we can prove that there is no subgraph of K_4^2 isomorphic to $S(2, 4)$ with constant corners: (By contradiction) Assume that $T(2, 4)$ is such a subgraph with corners 00, 11, 22, 33. The K_4 s that constitute the K_4 -covering of $T(2, 4)$ must lie on parallel lines (rows or columns) of K_4^2 . And each of those K_4 s contain exactly one corner. If the K_4 s are rows, then the exterior edges must lie in the columns (or *vice versa*). But with one of the vertices in each column being a corner (and not incident to an exterior edge) the 3 remaining vertices can only accommodate 1 edge. Four columns then yield at most 4 exterior edges but $T(2, 4) \simeq S(2, 4)$ has $\binom{4}{2} = 6$ exterior edges, a contradiction. The same argument shows that $T(2, m)$ does not exist for any even $m > 2$. The question of $T(n, m)$ for even $m > 2$ and $n > 2$ remains open. Also open is the question as to whether, for odd m , any other (nonisomorphic) embeddings with constant corners exist.

4.12. Tower of Hanoi on m Pegs. How to generalize the Tower of Hanoi puzzle to m pegs, $m > 3$? Obviously, if we add more pegs and retain the same rules, the puzzle just becomes easier to solve (although the minimum number of moves, even for $m = 4$ (and arbitrary n), has never been completely determined. The Frame-Stewart algorithm, which with 4 pegs takes about $\sqrt{2n+1}$ moves, is conjectured to be optimal (see Wikipedia, "Tower of Hanoi (Four pegs and beyond)". Scorer, Grundy & Smith [10] presented several variants. The one we found most amusing is called "Traveling Diplomats". The additional rules they propose for moving discs are described in terms of arcane diplomatic protocols for moving English diplomats from Praha to Geneva via two airlines that circulate (with flights in both directions) through 5 major capitols, Berlin, Praha, Rome, Geneva & London. So $m = 5$ and the number of diplomats, n , is arbitrary. However, diplomats are strictly ordered by rank and (per Scorer, Grundy & Smith)

- a:** "Each member must always travel three stages by air, for consultation at the two intermediate towns.
- b:** No member may start from, visit, or end up in a town at which one of his subordinates is stationed."

How was the transfer most quickly done?"

The routes of the two airlines are presented by a "map" but there is an equivalent (and more helpful) diagram in Figure 5.

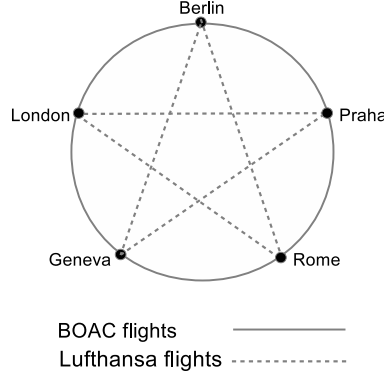


Figure 5-Map of airline routes

With this map it is easy to see that

- (1) Any move, from town X to town Y, can be made in 3 stages on a unique airline.
- (2) There is just one town unvisited at each move, so by Rule **b**, all subordinates of the transient diplomat must be at that (unvisited)town.

From this one may deduce that a move from ik^{n-1} to jk^{n-1} , $i \neq j$ can be made iff $k = 2^{-1}(i + j)$ which means that the graph of the Traveling Diplomats puzzle is $T(n, 5)$. Scorer, Grundy & Smith also noted that the rules as they state them could be extended to $T(n, p)$ for any prime p .

What other extension of the Towers of Hanoi might there be? The conditions that seem necessary to us are

- a:** The graph should be isomorphic to $S(n, m)$,
- b:** It should be a subgraph of $K(n, m)$, and
- c:** It should have constant corners.

We have constructed such graphs, $T(n, m)$, for all n if m is odd, and shown they do not exist for $n = 2$ if $m > 2$ is even. In our opinion, any such $T(n, m)$ defines a generalized Towers of Hanoi puzzle: A disc may be moved from peg i iff all smaller discs are on peg $k \neq i$ and then it may be moved to peg $j = (2k - i) \bmod m$. It follows then that $j \neq k$ and $k = 2^{-1}(i + j) \bmod m$, making the graph of the game $T(n, m)$.

Since we have altered τ here, replacing $2 \bmod 3$ by $2^{-1} \bmod m$ for any odd m , the calculation of the components for $\tau : S(n, m) \rightarrow T(n, m)$ will look a little different:

Theorem 8. $t_n = \tau_n(v_1, v_2, \dots, v_n) = \left(2^{-(n-1)}v_n + \sum_{i=1}^{n-1} 2^{-i}v_i \right) \bmod m$, so $t_n = 2^{n-1}\widetilde{v}_n \pmod{3}$.

And, also as before (Theorem 4),

Theorem 9. $v_n = \tau_n^{-1}(t_1, t_2, \dots, t_n) = \left(2^{n-1}t_n - \sum_{j=1}^{n-1} 2^{j-1}t_j \right) \bmod m$.

4.13. A New Beginning. The Tower of Hanoi puzzle, as proposed by Eduard Lucas in 1881, started with all n discs on one peg (say 1) and ended with all of them on another peg (say 0). Some puzzles, such as Rubik's Cube, are started from a random position. If the Tower of Hanoi is started from a random position, it would appear to be a much more complex problem.

Example 6. Let $n = 4$ and start at 1201. The goal is to reach $0^4 = 0000$ in the minimum number of moves.

It would be simple to select a position at random by labeling the sides of a triangular prism 0, 1, 2 and tossing it n times to generate a random member of $\{0, 1, 2\}^n$. In the example, starting at 1201, the largest disc starts on peg 1 and must be moved to peg 0. In order to do that the three smaller discs must be moved to peg 2. After the largest disc is transferred from peg 1 to peg 0, the three smaller discs may be transferred from peg 2 to peg 0. Thus the problem is recursively solvable and the same holds in general. However, deciding the optimal move from a given position, such as 1201 seems overwhelming.

Lemma 4. $\forall v \in S(n, m), \exists!$ minimum length path from v to 0^n . Its length is

$$\ell(v) = \sum_{i=1}^n \bar{v}_i 2^{n-i}$$

where

$$\bar{v}_i = \begin{cases} 0 & \text{if } v_i = 0 \\ 1 & \text{if } v_i \neq 0 \end{cases}.$$

Proof. By induction on $\ell(v)$. If $\ell(v) = 0$, $v = 0^n$ and the result is trivial. If true for $\ell(u) = \ell > 0$ and $\ell(v) = \sum_{i=1}^n \bar{v}_i 2^{n-i} = \ell + 1$ then there are two cases:

Case 1: $\bar{v}_n \neq 0 \Rightarrow v_n \neq 0$, so replace v_n by 0 to get $u \in V_{S(n, m)}$. By the definition of $S(n, m)$, $\{u, v\} \in E_{S(n, m)}$ and $\sum_{i=1}^n \bar{u}_i 2^{n-i} = \ell$. Thus the unique shortest path from v to 0^n starts with $\{u, v\}$ and continues with the unique shortest path from u to 0^n .

Case 2: $\bar{v}_n = 0 \Rightarrow v_n = 0 \Rightarrow \exists! h < n$ such that $v_h \neq 0$ and $v_j = 0$ for $j > h$. Let $u \in V_{S(n, m)}$ be such that

$$u_i = \begin{cases} v_i & \text{if } i < h \\ 0 & \text{if } i = h \\ v_h & \text{if } i > h \end{cases}.$$

Again $\{u, v\} \in E_{S(n, m)}$ and $\sum_{i=1}^n \bar{u}_i 2^{n-i} = \ell$ so we are done. □

Thus the antipodes of $0^n \in V_{S(n, m)}$ are those v such that $\forall i, v_i \neq 0$. There are $(m-1)^n$ of them, each at distance $2^n - 1$ from 0^n . Since $T(n)$ is isomorphic to $S(n, 3)$, we can use the correspondence $\tau : S(n, 3) \rightarrow T(n)$ and its inverse to solve the Tower of Hanoi problem with random starting position.

Example 7. For $n = 4$, $\tau(v_1, v_2, v_3, v_4) =$

$$(v_1, 2(v_1 + v_2) \bmod 3, 4(2v_1 + v_2 + v_3) \bmod 3, 8(4v_1 + 2v_2 + v_3 + v_4) \bmod 3)$$

in general so the matrix for τ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix}.$$

Surprisingly, this matrix is self-inverse, so it is also the matrix for $\tau^{-1}(t_1, t_2, t_3, t_4)$. Therefore

$$\tau^{-1}(1, 0, 2, 0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

$\ell(1, 2, 1, 0) = 1 \cdot 2^{4-1} + 1 \cdot 2^{4-2} + 1 \cdot 2^{4-3} + 0 \cdot 2^{4-4} = 14$. The vertex one step closer to 0^4 in $S(4, 3)$ is $(1, 2, 0, 1)$ and the one after that is $(1, 2, 0, 0)$. Continuing on in this way we generate the minimum length path from $(1, 2, 1, 0)$ to 0^4 in $S(4, 3)$. From there we apply τ to obtain the minimum length path from $(1, 0, 2, 0)$ to 0^4 in $T(4, 3)$:

ℓ	$S(4, 3)$	$T(4, 3)$
14	1210	1020
13	1201	1010
12	1200	1011
11	1022	1211
10	1020	1210
9	1002	1220
8	1000	1222
7	0111	0222
6	0110	0220
5	0101	0210
4	0100	0211
3	0011	0010
2	0010	0012
1	0001	0001
0	0000	0000

4.14. Another Formula. We can also write a formula to solve the Traveling Diplomats puzzle of [10]: Suppose we take $n = 4$ and number the capitols, Praha $\rightarrow 0$, Geneva $\rightarrow 1$, Paris $\rightarrow 2$, Rome $\rightarrow 3$, and London $\rightarrow 4$ in the order of the BOAC circuit. The challenge is to transport all four diplomats from Praha to Geneva, so we start at 0^4 and end at 1^4 on $T(4, 5)$. In $S(4, 5)$ the minimum path is $\eta^{-1}(\ell)$, $0 \leq \ell \leq 15$ and mapping that path to $T(4, 5)$ by τ , whose matrix of coefficients (mod 5) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 3 & 4 & 4 & 0 \\ 3 & 4 & 2 & 2 \end{pmatrix}$$

we get

ℓ	$S(4, 5)$	$T(4, 5)$
0	0000	0000
1	0001	0002
2	0010	0042
3	0011	0044
4	0100	0344
5	0101	0341
6	0110	0331
7	0111	0333
8	1000	1333
9	1001	1330
10	1010	1320
11	1011	1322
12	1100	1122
13	1101	1124
14	1110	1114
15	1111	1111

The matrix of coefficients (mod 5) for $\tau^{-1} : T(4, 5) \rightarrow S(4, 5)$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 2 & 0 & 0 \\ 4 & 3 & 4 & 0 \\ 4 & 3 & 1 & 3 \end{pmatrix}$$

which might be used to recall a random distribution of the four diplomats among the 5 capitols to back to Praha in the most efficient manner.

The nonzero entries of the matrices in these two examples have pseudorandom characteristics since they are sums of exponentials reduced mod m (see Wikipedia). It is amusing that they act to bring order out of the apparent chaos of moves to solve Tower of Hanoi problems. The key ingredient of this magic is the Scorer-Grundy-Smith transform, which takes the combinatorial structure of $T(m, n)$ and maps it onto the geometric structure of $S(m, n)$.

4.15. Is $\tau : S(4, 3) \rightarrow T(4, 3)$ Self-inverse by Coincidence or Principle? In Example 7 we noticed that $\tau = \tau^{-1}$. Is this just a coincidence or the result of some underlying principle which make it true for all n, m ? The answer is somewhere in between: If $m = 3$ it holds for all n . This can be seen from Theorems 8 & 9, since $2^{-1} = 2 = -1 \pmod{3}$. If $m = 5$ it does not hold for $n = 4$ (shown in Section 4.14) and seems unlikely for any n if $m > 3$.

4.16. Summing Up. The two main things accomplished in this paper are:

- (1) Answering the question of the title in the affirmative, and
- (2) Making explicit what was implicit in Scorer, Grundy & Smith's paper [10]. They described a "trilinear mapping" from $T(n)$ to $S(n, 3)$ in synthetic and qualitative terms. Utilizing the self-similarity of the Sierpinski graph, we characterized that mapping by a recurrence and solved the recurrence. What was synthetic and qualitative has become analytic, computational and surprisingly efficient.

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